Polyhedra of Small Order and Their Hamiltonian Properties

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Abstract

We describe the results of an enumeration of several classes of polyhedra. The enumerated classes include polyhedra with up to 12 vertices and up to 26 edges, simplicial polyhedra with up to 16 vertices, 4-connected polyhedra with up to 15 vertices, and bipartite polyhedra with up to 22 vertices.

The results of the enumeration were used to systematically search for certain minimal non-Hamiltonian polyhedra. In particular, the smallest polyhedra satisfying certain toughness-like properties are presented here, as are the smallest non-Hamiltonian, 3-connected, Delaunay tessellations and triangulations. Improved upper and lower bounds on the size of the smallest non-Hamiltonian, inscribable polyhedra are also given.

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1 Introduction

This paper describes an enumeration of several classes of polyhedra. It extends work done by many other researchers; references are given in the appropriate sections. Among the classes of polyhedra enumerated are:

- Polyhedra with up to 12 vertices, and up to 26 edges.
- Simplicial polyhedra with up to 16 vertices.
- 4-connected simplicial polyhedra with up to 17 vertices.
- 4-regular polyhedra with up to 22 vertices.
- 4-connected and minimally 4-valent polyhedra with up to 15 vertices.
- Bipartite polyhedra with up to 22 vertices.
- Non-Hamiltonian polyhedra with up to 14 vertices, 26 edges, or 12 faces.
- Non-Hamiltonian simplicial polyhedra with up to 17 vertices.

These results are discussed in Sections 3–5. Section 3 describes the enumeration of simplicial polyhedra. Section 4 describes the various approaches used to enumerate different classes of polyhedra, and contains enumeration of these classes. In particular, this section contains a refinement of Tutte's inductive definition of 3-connected planar graphs that may be of independent interest. It also contains an example illustrating why it may be difficult to find an inductive definition of the class of self-dual polyhedra. Section 5 focuses on the generation of non-Hamiltonian polyhedra.

One goal of this research was to find minimal examples of non-Hamiltonian polyhedra that satisfy certain additional graph-theoretical properties. The results are summarized here: the examples themselves, and the relevant definitions, appear in Section 6. The following list contains, for each combination of properties, the number of vertices in the smallest simplicial example, followed by the number of vertices and faces in the smallest polyhedral example:

- 1-tough, non-Hamiltonian polyhedron (13 vertices; 13 vertices, 10 faces)
- 1-supertough, not 1-Hamiltonian (10 vertices; 10 vertices, 8 faces)
- 1-supertough, not Hamiltonian (15 vertices; 15 vertices, 11 faces)

In Section 7, we turn our attention to inscribable polyhedra. We present enumerations of inscribable simplicial polyhedra with up to 14 vertices and circumscribable simplicial polyhedra with up to 16 vertices. We also exhibit the (unique) minimal polyhedron that is neither inscribable nor circumscribable; it has 10 vertices and is self-dual.

In Section 8, the results of the earlier sections are applied to the problem of finding minimal non-Hamiltonian Delaunay tessellations and triangulations. In particular, the smallest non-Hamiltonian, 3-connected, Delaunay tessellations have 13 vertices and 10 faces. There are 3 nonisomorphic graphs that have this property; one of these can be realized in several different ways, so there are actually 4 combinatorially distinct minimal examples. The smallest non-Hamiltonian, 3-connected Delaunay triangulations have 13 vertices and 21 faces (i.e., the nontriangular face is a quadrilateral); there are 2 nonisomorphic minimal examples.

Section 9 contains some results concerning minimal non-Hamiltonian inscribable polyhedra. Using the results of our enumeration, we have determined that the number of vertices in the smallest simplicial non-Hamiltonian inscribable polyhedra lies between 18 and 20, inclusive. There are at least 11 nonisomorphic simplicial non-Hamiltonian inscribable polyhedra with 20 vertices. In the nonsimplicial case, there are exactly three non-Hamiltonian inscribable polyhedra with 19 vertices and 13 faces; we conjecture that these are minimal.

2 Preliminaries

For the relevant background in combinatorial geometry and graph theory, see [22] and [5]. Throughout this paper, polyhedron means a 3-polyhedron. We make implicit use of Steinitz' theorem that a graph is realizable as a 3-polyhedron (*polyhedral*) if and only if it is planar and 3-connected. Two polyhedra are *combinatorially equivalent* if they are isomorphic; two polyhedral graphs embedded in the plane are combinatorially equivalent if they are isomorphic *and* the isomorphism preserves the identity of the outer face. A *stellation* of a graph G is any graph obtained by choosing a face f of G, inserting a new vertex inside f, and connecting the new vertex to all vertices of G on the boundary of f. If the new vertex is connected to some (but not necessarily all) of the boundary vertices of f, the resulting graph is called a *partial stellation* of G.

We use the following notation. $\mathcal{S}_n$ denotes the class of simplicial polyhedra with n vertices (i.e., polyhedra in which all faces are triangles.) A polyhedron with n vertices and k faces is called an (n,k)-polyhedron; the class of all (n,k)-polyhedra is denoted $\mathcal{P}_{n,k}$. The class of (n,k)-polyhedra is nonempty if and only if $n \leq 2k-4$ and $k \leq 2n-4$; if these inequalities are satisfied, we call (n,k) a *feasible pair*. We use $|\cdot|$ to denote cardinality, with the following conventions: if G is a graph, S is a point set, and $\mathcal{G}$ is a class of polyhedra (e.g., $\mathcal{S}_n$), then $|G|$, $|S|$, and $|\mathcal{G}|$ denote, respectively, the number of vertices in G, the number of elements in S, and the number of distinct combinatorial types in $\mathcal{G}$.

3 Generating simplicial polyhedra

The fundamental operation needed to generate $\mathcal{S}_n$, the simplicial polyhedra with n vertices, is the operation $\mathtt{augment}(G, e_1, e_2)$ illustrated in Figure 1. It is defined as follows. Given two distinct oriented edges $e_1 = vw$ and $e_2 = vx$ with a common tail v, v is "stretched" into an edge uv, and edges uw and ux are added. All neighbors of v that are between w and x (moving counterclockwise about v) are then disconnected from v

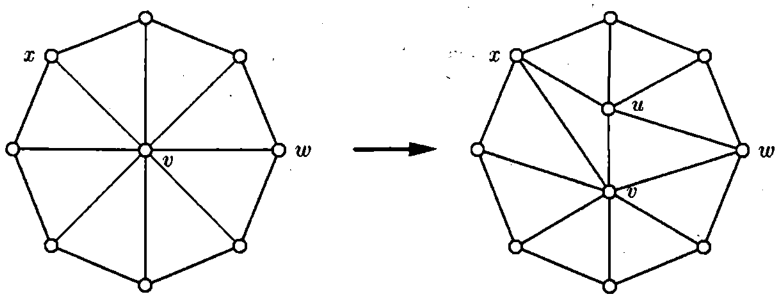

Figure 1: The "vertex stretching" operation used to generate simplicial polyhedra.

and attached to u. (Note that there may not be any such neighbors, in which case the new vertex u will have degree 3.) It is well known (see, for example, [6]), that for $n \geq 4$, $\mathcal{S}_n$ can be generated by applying the **augment** operation to every member of $\mathcal{S}_{n-1}$ in every possible way and checking for duplicates, and that $\mathcal{S}_4$ consists of a single graph (the tetrahedron).

The procedure outlined in the preceding paragraph causes many redundant candidate graphs to be generated. The total number of candidate graphs generated can be considerably reduced by applying a few simple observations.

1. Simplicial polyhedra with minimum degree 4 or 5 can be efficiently generated using an inductive procedure defined in [3], so it is only necessary to generate candidate graphs with minimum degree 3.

2. In view of Observation 1, it is only necessary to apply the operation **augment**($e1, e2$) in situations where e_1 and e_2 are adjacent edges. Indeed, it is shown in [6] that any simplicial polyhedron with degree 3 can be generated by performing the **augment** operation on some pair of incident edges in some graph in $\mathcal{S}_{n-1}$.

3. Since e_1 and e_2 play symmetric roles in the definition of **augment**(e_1, e_2), it is only necessary to apply the operation when e_2 is the clockwise neighbor of e_1 about their common tail.

4. Define two oriented edges e and e' in graph G to be *automorphism-equivalent* if there is an isomorphism of G onto itself mapping the tail and head of e onto the tail and head (respectively) of e'. This relation partitions the oriented edges of e into *automorphism-equivalence* classes. The preceding observations imply that for each base graph, it is adequate to choose one oriented edge e_1 from each automorphism-equivalence class, let e_2 be the clockwise neighbor about its tail, and to then apply **augment**(e_1, e_2) only to pairs of oriented edges e_1 and e_2 constructed in this fashion. An efficient and practical procedure for constructing the automorphism-equivalence classes for polyhedral graphs is described in [26].

In addition to the preceding optimizations, several implementation details are worth noting. Duplicate graph detection could be performed using the isomorphism-testing algorithm of [26], which is based on the partitioning of oriented edges into automorphism equivalence classes mentioned above. However, the simple isomorphism-testing algorithm described in [6], while asymptotically slower than the algorithm of [26], appears to be significantly faster for the small values of n relevant to this paper.

Efficient searching for possible duplicates can be done using standard chain-bucket hashing techniques [29]. The following hash function $h(\cdot)$ is invariant under isomorphism, can be computed rapidly, and seems to have nice distribution properties. Let G be a simplicial polyhedron with n vertices. For each vertex v_i, let $s(v_i)$ be the sum of the squares of the degrees of the neighbors of v_i. Let $s_1 \ldots s_n$ be the n values of $s(v_i)$, sorted in ascending order. The hash function for G is then given by

$$h(G) = \sum_{i=1}^{n} s_i p^{i-1} \bmod q, \tag{3.1}$$

for suitably chosen primes p and q. In addition to its use for streamlining the search for duplicates, this hash function is useful for producing large catalogs when disk space is limited. Indeed, one can partition the range $0, 1, \ldots, q-1$ into k disjoint intervals and run the generating program k times, once for each interval, each time ignoring all candidate graphs that fall outside the appropriate interval.

The number of simplicial polyhedra of each order up to 16 is shown in Table 1. The third column shows the number of distinct degree sequences that are realized by simplicial polyhedra of the given order. The fourth column shows the number of simplicial polyhedra with minimum degree at least 4. As indicated above, these were separately generated by a program implementing Batagelj's inductive definition of this subclass [3]. The final column shows the number of 4-connected simplicial polyhedra. These were obtained by testing each minimally 4-valent simplicial polyhedron for 4-connectivity.

Counts of simplicial polyhedra with up to 11 vertices can be found in [22] (also, see [7].) Simplicial polyhedra with 12 vertices were first enumerated by Bowen and Fisk [6]. The values in the above table up to and including $n = 14$ have been independently confirmed by Warren Smith. The values in the fourth column were previously computed by Holton and McKay [25], and earlier by Hucher *et al.* for $n \leq 14$ [27].

4 Generating polyhedra

There are two different approaches to enumerating a class of polyhedra $\mathcal{P}_{n,k}$, which we call the subtractive and additive approaches. Both have their uses.

The *subtractive approach* generates $\mathcal{P}_{n,k}$ from $\mathcal{P}_{n,k+1}$ by systematically deleting each edge from each $P \in \mathcal{P}_{n,k+1}$, verifying that the resulting graph remains 3-connected, and checking for duplicates. Two simple improvements speed up the algorithm considerably: (1) deleting one edge from each automorphism equivalence class, and (2) generating candidate graphs only if the new face would be a maximum-valence face (since otherwise the same candidate graph will be generated from a different base graph.) Since $\mathcal{P}_{n,2n-4} = \mathcal{S}_n$, the subtractive approach can in principle be used to generate all polyhedra with n vertices once $\mathcal{S}_n$ has been computed.

n	Graphs	Sequences	Minimum degree ≥ 4	4-connected
3	1	1		1
4	1	1		1
5	1	1	1	1
6	2	2	1	1
7	5	5	1	1
8	14	13	2	2
9	50	33	5	4
10	233	85	12	10
11	1,249	199	34	25
12	7,595	445	130	87
13	49,566	947	525	313
14	339,722	1,909	2,472	1,357
15	2,406,841	3,713	12,400	6,244
16	17,490,241	7,006	65,619	30,926
17	?	?	357,504	158,428

Table 1: The number of nonisomorphic simplicial polyhedra and distinct maximal planar degree sequences for $n \leq 15$, and the number of nonisomorphic 4-valent and 4-connected simplicial polyhedra for $n \leq 17$.

The *additive approach* uses the theory of 3-connected graphs developed by Tutte [44]. Tutte defined two basic operations, called *face-splitting* and *vertex-splitting*. These two operations, which are illustrated in Figure 2, are dual to one another. The inverse operations are, respectively, called *face-merging* and *vertex-merging*. An edge in a 3-connected graph is called *removable* if deleting it (i.e., merging the two faces on either side of it) preserves 3-connectivity. An edge in a 3-connected graph is called *shrinkable* if shrinking it to a vertex (i.e., merging its two endpoints) would not create a *multiedge* (a pair of edges with the same two endpoints).

Tutte proved that if $n > 4$ and $k > 4$, any graph in $\mathcal{P}_{n,k}$, with one exception, can be obtained either by applying a face-splitting operation to a graph in $\mathcal{P}_{n,k-1}$ or by applying

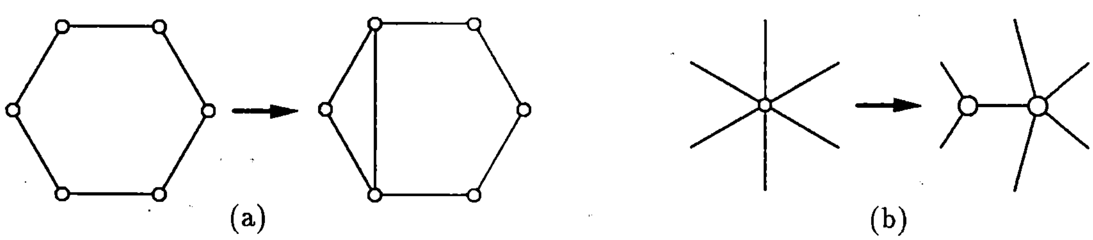

Figure 2: The two splitting operations. (a) Face-splitting: a face is split by adding an edge. (b) Vertex-splitting: a vertex is split, the two new vertices are joined, and the edges incident on the original vertex are apportioned between the two new vertices.

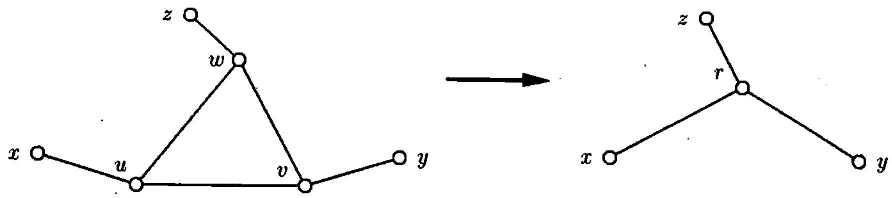

Figure 3: Case 1 in the proof of Theorem 4.1

a vertex-splitting operation to a graph in $\mathcal{P}_{n-1,k}$. The one exception is the wheel, W_n, consisting of $n-1$ vertices of degree 3 arranged in a cycle about a hub vertex of degree $n-1$. The following theorem refines Tutte's theorem by showing that, for fixed n and k, only one of these two operations need be performed:

Theorem 4.1 *If $k \geq n$, every graph in $\mathcal{P}_{n,k}$ (with the single exception of the wheel W_n if $n = k$, and otherwise without exception) can be generated by applying a face-splitting operation to some graph in $\mathcal{P}_{n,k-1}$.*

Proof Assume $n \leq k$, $G \in \mathcal{P}_{n,k}$, and G is not a wheel. Let m denote the number of edges in G. We show that G has a removable edge. The proof is by induction on n. For $n \leq 5$, the result is easily verified by inspection. Indeed, there are only three polyhedra with 5 or fewer vertices: the two wheels W_4 and W_5, and the triangular bipyramid, which has a removable edge.

For the induction step, we first note that since $k \geq n$, the average valence of a face is less that 4 (this follows easily from Euler's formula), so G has at least one triangular face. Let T be this face, and let u, v, and w be the three vertices on the boundary of T. There are 3 cases, depending on the number of boundary vertices that have degree 3.

<u>Case 1:</u> u, v, and w all have degree 3.

Let x, y, and z be, respectively, the neighbors of u, v, and w that are not in the triangle uvw. Notice that x,y, and z must all be distinct. Indeed, if they were all the same, then G would be the wheel W_4. If two of them were identical (say x and y), then removing x and w would separate uv from the rest of the graph, violating 3-connectivity.

Let G' be the graph obtained by collapsing uvw to a single vertex, r (see Figure 3). It is easy to verify that G' is 3-connected. Let k', n', and m' be, respectively, the number of faces, vertices, and edges of G'. We have $k' = k - 1$, $n' = n - 2$, and $m' = m - 3$. Hence $k' > n'$ (so, in particular, G' is not a wheel), and $m' < m$. So by the inductive assumption, G' has a removable edge, say e. The edge e cannot be incident on r, since $\texttt{degree}(r) = 3$. Hence e is a removable edge of G as well.

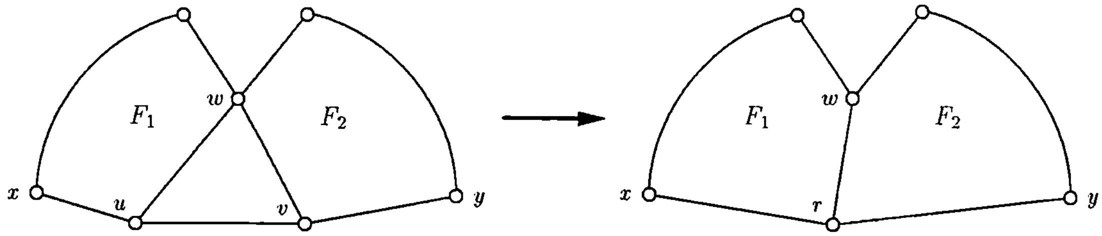

Figure 4: Case 3 in the proof of Theorem 4.1

Case 2: At least two of the three vertices (without loss of generality, assume u and v) are ≥ 4.

Result (3.4) in [44] (page 446) may be restated in our terminology as follows: if uvw is a triangle and neither edge uv nor uw is removable, then either u or v has degree 3.[1] It follows that either edge uv or uw is removable.

Case 3: Two vertices (say u and v) have degree 3, the remaining vertex (w) has degree > 3.

Let x (respectively, y) be the neighbor of u (respectively, v) that is not a boundary vertex of T. As in Case 1, x and y must be distinct. Let G' be the graph obtained by collapsing the edge uv to a single vertex r, with neighbors w, x, and y, as illustrated in Figure 4. We claim that G' is 3-connected. Assume for the moment that the claim is true. Let n', k', and m' be, respectively, the number of vertices, faces, and edges of G'. We have $n' = n - 1$, $k' = k - 1$, and $m' = m - 2$. In particular, $k' \geq n'$. Also, G' is not a wheel (since, if it were, G would be a wheel with hub w). By induction, G' has a removable edge e. Since r has degree 3, r cannot be an endpoint of e. It is not hard to see that e is also a removable edge of G.

It remains to show that G' is 3-connected. We must show that given any pair of distinct vertices in G', there are 3 vertex-disjoint paths between them. If w is not in the pair, this is straightforward to verify (since G is 3-connected). So assume one vertex is w, the other some vertex a. If $a = r$, the verification is again straightforward, so assume $a \neq r$. Let F_1 (respectively, F_2) be the face opposite the edge uw (respectively, vw) from T in G, and give the corresponding faces in G' the same names. Assume that wux forms part of a clockwise walk around the boundary of F_1 (see Figure 4). Since G is three-connected, there are three vertex-disjoint paths, Π_1, Π_2, and Π_3, from w to a in G. Assume that one of these paths (Π_1) uses the edge wu and another path (Π_2) uses the edge wv (otherwise, the paths correspond naturally to three disjoint paths from w to a in G' and the proof is complete).

Notice that Π_2 cannot contain a vertex on the boundary of F_1 other than w (or possibly a, if a is on the boundary of F_1). Indeed, suppose it contained a vertex $p \neq w$ on the boundary of F_1. Let H be the graph obtained from G by placing a new vertex z inside face F_1 and connecting it to p, w, and u. Then H is planar. But H would then

[1] Result (3.4) in [44] is actually stated in a weaker form, using the stronger hypothesis that neither uv nor uw is either removable or shrinkable. Nevertheless, Tutte's proof of this result uses only the assumption that neither uv nor uw is removable, so it may be stated in this stronger form.

have a set of 9 vertex-disjoint paths connecting each of p, u, and w to each of z, a, and v (namely: the three edges incident on z; the portion of Π_2 from p to a, the portion of Π_1 from u to a, and Π_3; and the portion of Π_2 from v to p and the edges uv and wv). Since a planar graph cannot contain a $K_{3,3}$ minor, this is impossible. Similarly, Π_1 cannot contain a vertex on the boundary of F_2 other than w (or possibly a). Also, since w has degree at least 4 and G is 3-connected, a cannot be simultaneously on the boundary of F_1 and the boundary of F_2.

Next, modify Π_3 as follows. Let b be the last vertex on the boundary of either F_1 or F_2 encountered by Π_3 from w to a. If $b = w$, do nothing. Otherwise, b cannot belong to both F_1 and F_2. If b belongs to F_1 (respectively, F_2), replace the portion of Π_3 from w to b with the arc of the boundary of F_1 (respectively, F_2) from w to b that does not contain u (respectively, v). After this modification, there is some $i \in \{1,2\}$ such that Π_3 does not contain any boundary vertex of F_i. Assume, without loss of generality, that $i = 1$. Let c be the last vertex on Π_1 that is also on the boundary of F_1. Since u has degree 3, $c \neq u$. Since neither Π_2 nor Π_3 contains any vertices of F_1 (other than w), we can "detour" Π_1 to go counterclockwise around the boundary of F_1 from w to c, thereby missing u. Now the modified paths $\{\Pi_i\}$ correspond exactly to 3 disjoint paths in G' from w to b (with r replacing v on Π_2). This proves the claim, and hence the theorem. ∎

Since vertex-splitting and face-splitting are dual to one another, we also have the following dual version of Theorem 4.1:

Theorem 4.2 *If $n \geq k$, every graph in $\mathcal{P}_{n,k}$ (with the single exception of the wheel W_n if $n = k$, and otherwise without exception) can be generated by applying a vertex-splitting operation to some graph in $\mathcal{P}_{n-1,k}$.*

Theorem 4.2 effectively halves the work required to generate $\mathcal{P}_{n,k}$ for $n > k$, since it says that we need only apply vertex-splitting to every vertex in $\mathcal{P}_{n-1,k}$. There is a natural correspondence between possible vertex-splitting operations and pairs of oriented edges (e_1, e_2) with a common tail. Hence the amount of work can be further reduced by choosing one oriented edge e_1 from each automorphism class and only applying the corresponding vertex-splitting operations. Also, it is only necessary to consider as candidates for e_2 the first half of the edges that have the same tail as e_1, moving clockwise about e_1. This is because the remaining pairs will be encountered with e_2 and e_1 playing opposite roles. Notice that it is important for efficiency reasons to modify the hash function of Section 3 to take face valences into account. We used the following modification of (3.1):

$$h(G) = \sum_{j=1}^{m} s_j p^{j-1} \bmod q.$$

Here, for each edge, we compute the sum of the squares of the degrees of the two endpoints and a small multiple (we used 5) of the squares of the valences of the two faces incident on the edge. The s_j's are these m computed values, sorted into ascending order.

In the special case of generating the "diagonal" entries $\mathcal{P}_{n,n}$ from $\mathcal{P}_{n-1,n}$, further saving of work is possible. It follows from Theorem 4.2 that for any $G \in \mathcal{P}_{n,n} - \{W_n\}$, both

G and its dual G^* will be generated. Hence, we introduce the notion of a *representative* of each dual pair. Whenever we generate a candidate graph G, we determine whether it is the representative. If it is the representative, we check whether it is a duplicate and proceed accordingly; if it is not the representative, we eliminate it immediately. This scheme saves disk space (since we only have to store one representative of each dual pair) and work (since we save roughly half the checks for duplicates).

To implement the representative scheme, we introduce a 2-variable selection function, $s(x,y)$, with the properties that (1) the value of $s(x,y)$ is always either x or y, and (2) $s(x,y) = s(y,x)$. Given a graph G for which $h(G) \neq h(G^*)$, we say G is the representative of the pair if and only if $s(h(G), h(G^*)) = h(G)$. (Here $h(\cdot)$ is the hash function.) A more precise description is as follows. For each candidate graph G, we compute $h(G)$ and $h(G^*)$. If $h(G) \neq h(G^*)$, we determine whether G is the representative; if so, we check whether G is a duplicate, otherwise we discard it immediately. If $h(G) = h(G^*)$, we check whether G is a duplicate; if it is not, we check whether G^* is a duplicate.

Notice that if G is self-dual, we will only do the second check the first time that G appears as a candidate graph. Otherwise, we do two duplicate checks only in the (rare) case where G is not self-dual but $h(G) = h(G^*)$. Notice also that the selection function should be chosen so as not to skew the uniform distribution of the hash function; for example, $s(x,y) = \max(x,y)$ would be a bad choice. In our implementation, we chose

$$s(x,y) = \begin{cases} \max(x,y) & \text{if } x+y \bmod r \text{ is even} \\ \min(x,y) & \text{otherwise} \end{cases}$$

for a large prime r.

Table 2 contains the values of $|\mathcal{P}_{n,k}|$ for all feasible pairs with $n \leq 12$, and for selected values with $n \leq 15$. Question marks indicate unknown values, blank entries indicate infeasible pairs. With the exception of $|\mathcal{P}_{16,28}| = 17,490,241$, not shown because of space, the table implicitly contains all known values (since $|\mathcal{P}_{n,k}| = |\mathcal{P}_{k,n}|$). Values for $n \leq 9$ were first published in [21]. Values for $n \leq 10$ and for $(11,11)$, $(11,12)$, $(11,13)$, and $(12,12)$ first appeared in [18]. The remaining values for $n \leq 11$ were first published in [20]. All other values appearing in Table 2 are new; the values for $(13,13)$ and $(14,13)$ have been independently discovered by Duijvestijn [17].

Table 3 contains the number of polyhedra with m edges for all $m \leq 26$. The values for $m \leq 22$ were published in [18], and the values for $23 \leq m \leq 25$ were independently discovered by Duijvestijn [17]. The value for $m = 26$ appears here for the first time.

We now present enumerations of certain subsets of $\mathcal{P}_{n,k}$. Table 4 shows the number of 4-regular polyhedra with 22 or fewer vertices. These were generated using the inductive algorithm given in [4]. By Euler's formula, a 4-regular polyhedron with n vertices has exactly $n+2$ faces. The third column of Table 4 contains the number of 4-connected, 4-regular polyhedra with n vertices.

Table 5 shows the number of (n,k)-polyhedra in which every vertex has degree at least 4 for $n \leq 15$. Each column was generated by applying the subtractive method, starting with the minimally-4-valent simplicial polyhedra with n vertices. Notice that applying the subtractive method is clearly valid (since a polyhedron obtained by adding

	n											
k	4	5	6	7	8	9	10	11	12	13	14	15
4	1											
5		1	1									
6		1	2	2	2							
7			2	8	11	8	5					
8			2	11	42	74	76	38	14			
9				8	74	296	633	768	558	219	50	
10				5	76	633	2,635	6,134	8,822	7,916	4,442	1,404
11					38	768	6,134	25,626	64,439	104,213	112,082	79,773
12					14	558	8,822	64,439	268,394	709,302	1,263,032	1,556,952
13						219	7,916	104,213	709,302	2,937,495	8,085,725	15,535,572
14						50	4,442	112,082	1,263,032	8,085,725	33,310,618	?
15							1,404	79,773	1,556,952	15,535,572	?	?
16							233	36,528	1,338,853	?	?	?
17								9,714	789,749	?	?	?
18								1,249	306,470	?	?	?
19									70,454	7,706,577	?	?
20									7,595	2,599,554	?	?
21										527,235	?	?
22										49,566	?	?
23											4,037,671	?
24											339,722	?
25												?
26												2,406,841
total	1	2	7	34	257	2,606	32,300	440,564	6,384,634	?	?	?
self-dual	1	1	2	6	16	50	165	554	1,908	6,667	23,556	?

Table 2: Values of $|\mathcal{P}_{n,k}|$, the number of nonisomorphic polyhedral graphs having n vertices and k faces.

m	total	self-dual	dual pairs
6	1	1	1
7	0		0
8	1	1	1
9	2		1
10	2	2	2
11	4		2
12	12	6	9
13	22		11
14	58	16	37
15	158		79
16	448	50	249
17	1,342		671
18	4,199	165	2,182
19	13,384		6,692
20	43,708	554	22,131
21	144,810		72,405
22	485,704	1,908	243,806
23	1,645,576		822,788
24	5,623,571	6,667	2,815,119
25	19,358,410		9,679,205
26	67,078,896	23,556	33,527,670

Table 3: Number of polyhedra with up to 26 edges.

n	4-regular	4-connected
6	1	1
7	0	0
8	1	1
9	1	0
10	3	3
11	3	1
12	11	8
13	18	7
14	58	37
15	139	55
16	451	220
17	1,326	499
18	4,461	1,862
19	14,554	5,174
20	49,957	18,258
21	171,159	57,107
22	598,102	198,474

Table 4: Number of 4-regular polyhedra with up to 22 vertices.

	n												
k	4	5	6	7	8	9	10	11	12	13	14	15	total
4	0												0
5		0	0										0
6		0	0	0	0								0
7			0	0	0	0	0						0
8			1	0	0	0	0	0	0				1
9				0	0	0	0	0	0	0	0		0
10				1	1	0	0	0	0	0	0	0	2
11					1	1	0	0	0	0	0	0	2
12					2	4	3	0	0	0	0	0	9
13						4	10	3	0	0	0	0	17
14						5	25	36	11	0	0	0	77
15							17	107	119	18	0	0	261
16							12	159	580	456	58	0	1,265
17								89	1,095	2,815	1,714	139	5,852
18								34	1,089	7,562	14,102	6,678	
19									491	10,096	47,890	67,651	
20									130	7,485	85,805	288,534	
21										2,806	87,124	651,596	
22										525	51,844	870,969	
23											16,534	712,861	
24											2,472	355,286	
25												98,587	
26												12,400	
total	0	0	1	1	4	14	67	428	3,515	31,763	307,543	3,064,701	

Table 5: The number of nonisomorphic polyhedral graphs having n vertices, k faces, and minimum degree at least 4.

an edge to a polyhedron with minimum degree 4 also has minimum degree 4), but the additive method may not be. Table 6 shows the number of 4-connected, (n, k)-polyhedra for $n \leq 15$. It was generated by testing each polyhedron with minimum degree 4 for 4-connectivity. (Notice that the tetrahedron is a special case: it is 4-connected but has minimum degree 3.)

Table 7 shows the number of bipartite (n, k)-polyhedra for $n \leq 22$. The values for each fixed n were computed by starting with the set of 4-regular polyhedra with $n-2$ vertices (and n faces), computing their duals, and then applying the subtractive method. This is valid because it is always possible to add edges to any bipartite polyhedron to obtain a quadrangulation.

Table 8 shows the number of *irreducible* polyhedra, which we define to be those polyhedra that do not have a removable edge. (In other words, these are wheels plus the counterexamples to the statement obtained by substituting "$k < n$" in Theorem 4.1.) The irreducible polyhedra with n vertices and k faces were generated by filtering the

	n												
k	4	5	6	7	8	9	10	11	12	13	14	15	total
4	1												1
5		0	0										0
6		0	0	0	0								0
7			0	0	0	0	0						0
8			1	0	0	0	0	0	0				1
9				0	0	0	0	0	0	0	0		0
10				1	1	0	0	0	0	0	0	0	2
11					1	0	0	0	0	0	0	0	1
12					2	3	3	0	0	0	0	0	8
13						3	7	1	0	0	0	0	11
14						4	20	24	8	0	0	0	56
15							13	70	70	7	0	0	160
16							10	112	366	252	37	0	777
17								60	686	1,591	867	55	3,259
18								25	700	4,416	7,497	3,207	
19									307	5,897	25,912	33,539	
20									87	4,401	47,030	146,823	
21										1,616	47,640	335,055	
22										313	28,289	449,468	
23											8,875	366,007	
24											1,357	181,118	
25												49,504	
26												6,244	
total	1	0	1	1	4	10	53	292	2,224	18,493	167,504	1,571,020	

Table 6: The number of nonisomorphic 4-connected polyhedral graphs with n vertices, k faces.

	k															
n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	total
8	1															1
9		0														0
10		0	1													1
11			0	1												1
12			1	1	3											5
13				0	2	3										5
14				1	3	6	11									21
15					0	7	15	18								40
16					2	9	41	66	58							176
17						0	24	125	212	139						500
18						2	27	178	602	793	451					2,053
19							0	106	807	2,400	2,893	1,326				7,532
20							8	92	958	4,482	10,407	10,798	4,461			31,206
21								0	420	4,964	21,454	42,992	40,168	14,554		124,552
22								8	322	4,554	31,259	104,549	180,123	150,560	49,957	521,332
total	1	0	2	3	10	27	126	593								

Table 7: The number of nonisomorphic bipartite polyhedral graphs having n vertices and k faces.

collection $\mathcal{P}_{n,k}$. It is an open question whether there is a more efficient way of generating them.

We conclude this section by mentioning self-dual polyhedra. Let $\mathcal{D}_n$ denote the collection of all self-dual polyhedra with n vertices. The values of $|\mathcal{D}_n|$, for $n \leq 14$, appear in Table 2. They were enumerated by filtering the collection $\mathcal{P}_{n,n}$.

It seems plausible that there should be a way of inductively generating the class $\mathcal{D}_{n+1}$ from the class $\mathcal{D}_n$ without necessarily having the entire collection $\mathcal{P}_{n,n}$ available. However, it is not possible to compute $\mathcal{D}_{n+1}$ by applying a vertex-splitting operation plus a face-splitting operations in every possible way to every graph in $\mathcal{D}_n$ and then filtering for self-dual polyhedra. The graph G of Figure 5 is an example of a graph in $\mathcal{D}_{13}$ that cannot be generated in this fashion from a graph in $\mathcal{D}_{12}$. Indeed, there are only 2 isomorphisms of G onto its dual. One isomorphism, $i_1(\cdot)$, takes vertex A onto face a, vertex A' onto face a', vertex B onto face b, etc. The other isomorphism, $i_2(\cdot)$, takes A onto face a', vertex A' onto face a, vertex B onto face b', etc. Now consider edge AB, and isomorphism $i_1(\cdot)$. Suppose there were a way to transform G into a graph in $\mathcal{D}_{12}$ by removing edge AB and shrinking some edge e of G in such a way that the isomorphism $i_1(\cdot)$ was preserved. On the one hand, if AB is removed from G, preserving $i_1(\cdot)$ requires removing edge ab from G^*, which is equivalent to choosing $e = CD$. On the other hand, removing AB from G is equivalent to shrinking edge $c'd'$ in G^*, so preserving $i_1(\cdot)$ requires choosing $e = C'D'$. These conflicting requirements show that it is impossible to remove edge AB and shrink some other edge to preserve $i_1(\cdot)$. The same argument works for any removable edge in G, and for $i_2(\cdot)$ as well as $i_1(\cdot)$. G is the only self-dual

	k											
n	4	5	6	7	8	9	10	11	12	13	14	total
4	1											1
5		1	0									1
6		1	1	0	0							2
7			2	1	0	0	0					3
8			2	6	1	0	0	0	0			9
9				8	10	1	0	0	0	0	0	19
10				5	44	21	1	0	0	0	0	71
11					38	173	37	1	0	0	0	249
12					14	362	607	74	1	0	0	1,058
13						219	2,348	1,999	138	1	0	4,705
14						50	3,073	12,611	6,370	275	1	22,380
15							1,404	28,885	58,753	20,025	?	
16							233	26,698	209,516	?	?	
17								9,714	329,165	?	?	
18								1,249	232,981	?	?	
19									70,454	3,569,749	?	
20									7,595	2,038,206	?	
21										527,235	?	
22										49,566	?	
total	1	2	5	20	107	826	7,703	81,231	914,973			

Table 8: The number of nonisomorphic irreducible polyhedral graphs having n vertices and k faces.

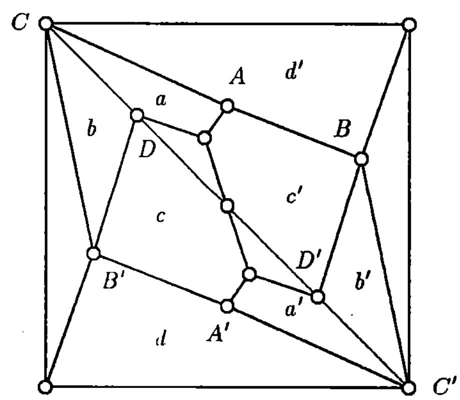

Figure 5: A self-dual graph with 13 vertices that cannot be obtained from a self-dual graph with 12 vertices by one vertex split and one face split.

n	Non-Hamiltonian Simplicial Polyhedra	Primitive Non-Hamiltonian Simplicial Polyhedra	Non-Hamiltonian 1-Tough Simplicial Polyhedra	Non-Hamiltonian 1-Supertough Simplicial Polyhedra
11	1	1	0	0
12	2	0	0	0
13	30	5	1	0
14	239	0	6	0
15	2,369	32	72	1
16	22,039	0	847	4
17	205,663	227	9,801	58

Table 9: Counts of nonisomorphic, non-Hamiltonian simplicial polyhedra.

graph with 13 or fewer vertices that has this property.

5 Generating Non-Hamiltonian Polyhedra

Table 9 contains the number of non-Hamiltonian simplicial polyhedra for $n \geq 17$. Barnette and Jucocovič have shown [2] that the count is 0 for $n < 11$. The values for $n \leq 16$ were obtained by filtering $\mathcal{S}_n$. The value for $n = 17$ was obtained by applying the **augment** operation of Section 3 to each polyhedron in $\mathcal{S}_{16}$ (using the optimizations discussed in Section 3, but only keeping the candidate graphs that are not duplicates and are also non-Hamiltonian).

We define a non-Hamiltonian simplicial polyhedron with n vertices to be *imprimitive* if it can be obtained from some *non-Hamiltonian* simplicial polyhedron by an application of the **augment** operation. A non-Hamiltonian simplicial polyhedron is *primitive* if it cannot be so generated (i.e., if any polyhedron obtained by performing the inverse of the vertex-stretching operation shown in Figure 1 is Hamiltonian). The primitive non-Hamiltonian simplicial polyhedra are counted in the third column of Table 9. The following conjecture is suggested by our observations for $n \leq 17$:

Conjecture 5.1 *Every primitive non-Hamiltonian simplicial polyhedron has an odd number of vertices.*

We note that Observation 2, from the list in Section 3, no longer holds when computing imprimitive simplicial polyhedra. The last two columns in Table 9 are discussed in the next section.

Table 10 contains the number of non-Hamiltonian polyhedra with n vertices and k faces for all $n \leq 14$, all $k \leq 12$, and a few selected other values. Holton and McKay have shown that there are no non-Hamiltonian trivalent polyhedra with $n < 38$ [25]; these

	n														
k	11	12	13	14	15	16	17	18	19	20	21	22	23	24	total
8	0	0													0
9	1	0	0	0	0										1
10	2	2	4	0	0	0									8
11	6	17	46	40	26	0	0	0							135
12	12	72	318	645	808	519	183	0	0	0					2,557
13	16	188	1,371	4,885	10,857			6,100	1,524	0	0	0			
14	16	326	3,783	21,844									0	0	
15	12	390	7,245	64,651											
16	6	326	9,857	134,552											
17	2	188	9,677	203,772											
18	1	72	6,843	228,238											
19		17	3,413	189,592											
20		2	1,157	115,642											
21			240	50,449											
22			30	14,948											
23				2,711											
24				239											
25															
26					22,039										
total	74	1,600	43,984	1,032,208											

Table 10: The number of nonisomorphic non-Hamiltonian polyhedral graphs having n vertices and k faces.

zero values are not all reflected in the table. Also, the values for $(16, 28)$ and $(17, 30)$, which are not shown in Table 10, appear in Table 9.

The non-Hamiltonian (n, k)-polyhedra were computed using a combination of methods. The entries for which the $\mathcal{P}_{n,k}$ had been generated were computed by filtering $\mathcal{P}_{n,k}$. The $(18, 13)$ entry was computed by applying the subtractive method (actually its dual, based on edge-shrinking) to all polyhedra in $\mathcal{P}_{19,13}$, and then saving only those polyhedra that are non-Hamiltonian and not duplicates. The values with $n = 13$ and $n = 14$ for $k \geq n$ were computed by starting with the (n, n) entry and then applying face-splitting, filtering for non-Hamiltonian graphs, and eliminating duplicates.

It is not, in general, possible to compute all non-Hamiltonian (n, k)-polyhedra by starting with all non-Hamiltonian simplicial polyhedra with n vertices, applying edge removal, and filtering for non-Hamiltonicity and 3-connectedness. The problem is that there exist nonsimplicial, non-Hamiltonian (n, k)-polyhedra with the property that adding any edge makes the polyhedron Hamiltonian. Examples of such polyhedra with 19 vertices and 33 faces are given in Section 9. We do not know if these are the smallest such examples.

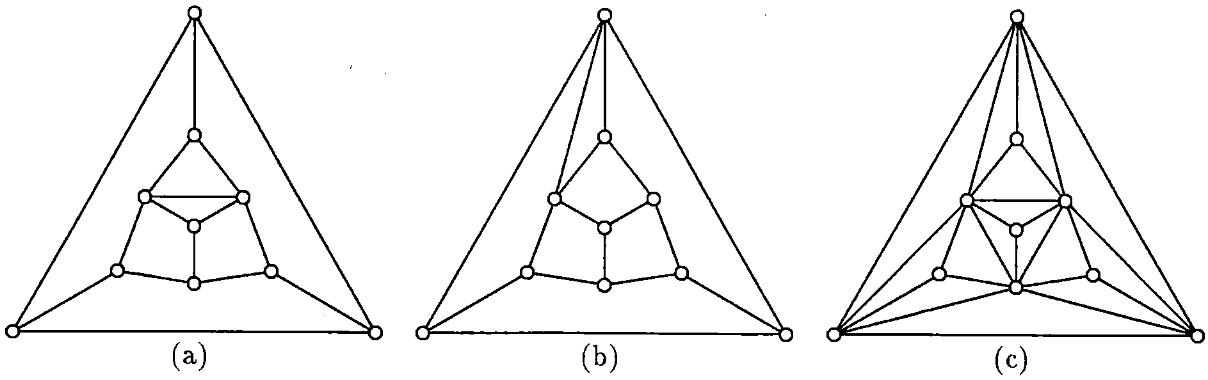

Figure 6: Three smallest 1-supertough graphs that fail to be 1-Hamiltonian. (a) and (b) are the two smallest polyhedra with this property, while (c) is the smallest simplicial polyhedron with this property.

6 Some Minimum Non-Hamiltonian Polyhedra

Using the generated catalogs of polyhedra discussed above, we were able to find several minimal examples of polyhedra with interesting Hamiltonian properties. We present them here without proofs.

A graph is k-Hamiltonian if deleting any k vertices leaves a Hamiltonian graph. Thomassen has given an example of a planar graph with 105 vertices that is 1-Hamiltonian but not Hamiltonian. It is shown in [14] that for $k > 1$, any k-Hamiltonian planar graph is $(k-1)$-Hamiltonian (note that $k = 2$ and $k = 3$ are the only non-vacuous cases).

A graph is *1-tough* if $c(G - S) \leq |S|$ for all nonempty $S \subseteq V(G)$. Here $G - S$ denotes the graph obtained by deleting S and all incident edges from G, and $c(\cdot)$ denotes the number of components. A graph is *1-supertough* if deleting any vertex leaves a 1-tough graph. Any 1-tough graph is 2-connected, and any 1-supertough graph is 3-connected (so a planar, 1-supertough graph is polyhedral).

The notion of toughness of a graph was originally defined by Chvátal as a weak form of Hamiltonicity [8]. It is noted in [8] that any Hamiltonian graph is 1-tough. It follows immediately that any 1-Hamiltonian graph is 1-supertough, and hence that any 1-Hamiltonian graph is 1-tough. The converses of these statements do not hold; here, we give minimal counterexamples for polyhedra and simplicial polyhedra.

The smallest polyhedron that is not 1-Hamiltonian is the cube, and the smallest simplicial polyhedron that is not 1-Hamiltonian is the fully stellated tetrahedron. Both these graphs are 1-tough, but not 1-supertough.

There are two nonisomorphic smallest 1-supertough planar graphs that fail to be 1-Hamiltonian. They have 10 vertices and 8 faces, and are shown in Figure 6(a) and (b). The (unique) smallest 1-supertough simplicial polyhedron that fails to be 1-Hamiltonian is shown in Figure 6(c); it also has 10 vertices.

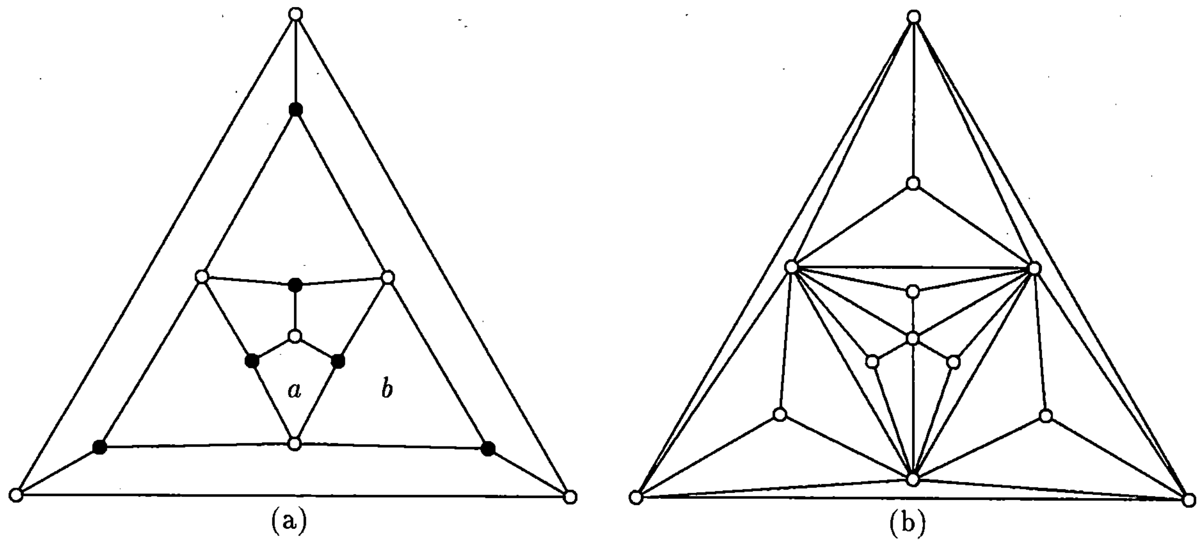

Figure 7: (a) The smallest 1-tough, non-Hamiltonian, polyhedron. (b) The smallest 1-tough, non-Hamiltonian, simplicial polyhedron.

The smallest 1-tough, non-Hamiltonian polyhedron is the 13-vertex, 10-face example shown in Figure 7(a). The significance of the markings on the figure will become apparent in Section 8. The smallest 1-tough, non-Hamiltonian simplicial polyhedron is the 13-vertex graph shown in Figure 7(b). Previously, Nishizeki gave a 19-vertex example [31].

The *shortness exponent* of a class of graphs was introduced in [23] as a measure of the non-Hamiltonicity of the class. Let $h(G)$ denote the length of the longest cycle in a graph. Then for any class $\mathcal{T}$ of graphs, the shortness exponent is defined by

$$\sigma(\mathcal{T}) = \liminf_{n\to\infty} \frac{\log h(G_n)}{\log |G_n|},$$

where the lim inf is taken over all sequences of graphs in $\mathcal{T}$ for which $|G_n| \to \infty$. By applying the construction of [12] to the graph of Figure 7(b), it can be shown that the shortness exponent of the class of 1-tough simplicial polyhedra is at most $\log_5 6$. This improves the bound of $\log_6 7$ given in [12].

The smallest 1-supertough, non-Hamiltonian planar graph has 15 vertices and 11 faces. It is shown in Figure 8(a). The smallest 1-supertough, non-Hamiltonian polyhedron has 15 vertices, and is shown in Figure 8(b). The 1-tough and 1-supertough non-Hamiltonian, simplicial polyhedra with up to 17 vertices are enumerated in the last two columns of Table 9.

The structure of the simplicial examples described in this section becomes clearer if we look at the "building blocks" of Figure 9. The minimal simplicial graph that is not 1-supertough (the stellated tetrahedron) is obtained by stellating the outer face of Figure 9(a). The minimal simplicial graph that is 1-supertough but not 1-Hamiltonian

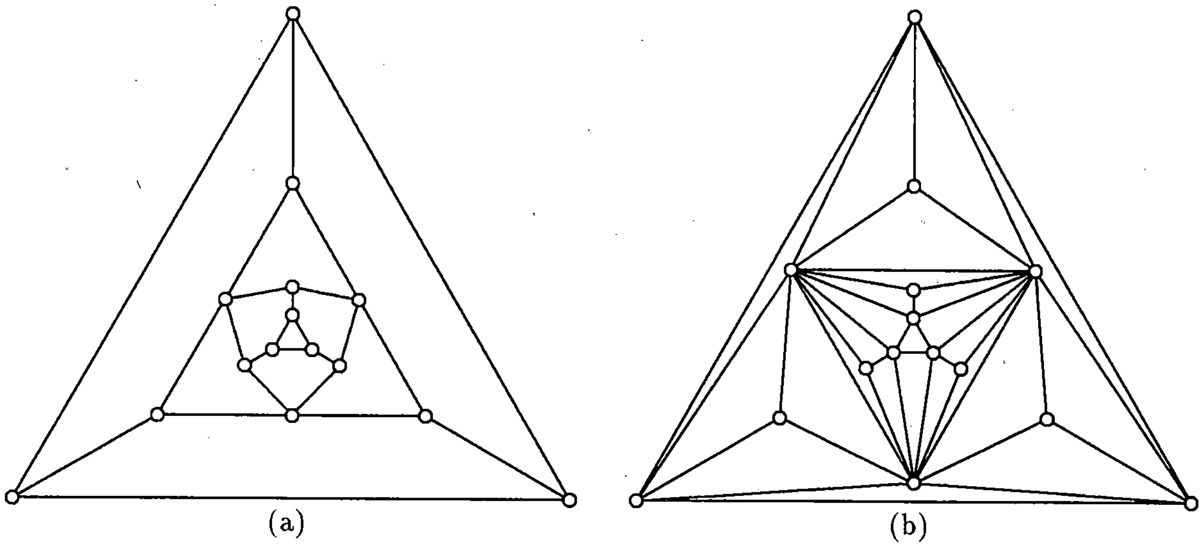

Figure 8: (a) The smallest 1-supertough, non-Hamiltonian, planar graph. (b) The smallest 1-supertough, non-Hamiltonian, simplicial polyhedron.

(Figure 6(c)) is obtained by stellating the outer face of Figure 9(b). The 11-vertex minimal non-Hamiltonian simplicial polyhedron of [2] consists of two copies of the graph of Figure 9(a), pasted together along a common face. The graphs of Figure 7(b) and Figure 8(b) consist, respectively, of one copy of Figure 9(a) and one of Figure 9(b), and two copies of Figure 9(b), pasted together along a common face.

We have given minimal examples of planar 3-connected graphs, both simplicial and non-simplicial, which remain 1-tough when j vertices are removed but fail to be k-Hamiltonian for $j, k \in \{0, 1\}$. The next logical class of graphs to consider in the progression starting with 1-tough and 1-supertough would be those graphs that remain 1-tough when two vertices are removed. However, it is shown in [13] that the planar graphs with this property are exactly the 4-connected planar graphs. These graphs are Hamil-

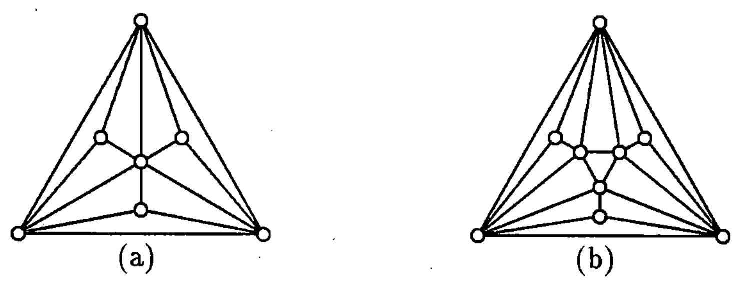

Figure 9: Two building blocks for non-Hamiltonian simplicial polyhedra.

tonian [43, 45] and 1-Hamiltonian (see [42]). M. D. Plummer has conjectured that all 4-connected planar graphs are 2-Hamiltonian [33].

7 Inscribable graphs and Delaunay Tessellations

A polyhedron is *inscribable* if it has a (combinatorially equivalent) realization as the edges and vertices of the convex hull of a noncoplanar set of points on the surface of a sphere in 3-space. A polyhedron is *circumscribable* if it has a (combinatorially equivalent) realization as a polyhedron each of whose faces is tangent to a common sphere. It is shown in [22] that a polyhedron is circumscribable if and only if its dual is inscribable.

A *Delaunay tessellation* is a 2-connected plane graph such that (1) the boundary vertices of the outer face are exactly the vertices of the convex hull; (2) the boundary vertices of every interior face are cocircular; and (3) no circumcircle about a face contains any vertices in its interior. A *Delaunay triangulation* is a Delaunay tessellation in which all interior faces are triangles and the boundary vertices of the outer face are exactly the extreme points of the vertex set. For a more conventional definition of Delaunay triangulations and tessellations as duals of Voronoi diagrams, and for a systematic exposition of their fundamental properties, see [1, 19, 34]. The word *nondegenerate* is sometimes used to distinguish Delaunay triangulations as we have defined them here. (A *degenerate* Delaunay triangulation is a triangulation obtained by adding edges to a Delaunay tessellation that is not a Delaunay triangulation.)

We state without proof several results about inscribable polyhedra, Delaunay triangulations, and the relations between them:

Lemma 7.1 ([24, 35, 36, 38]) *A polyhedron is inscribable if and only if weights w can be assigned to its edges such that:*

(W1) *For each edge e, $0 < w(e) < 1/2$.*

(W2) *For each vertex v, the total weight of all edges incident on v is equal to 1.*

(W3) *For each noncoterminous cutset $C \subsetneq E(G)$, the total weight of all edges in C is strictly greater than 1.*

Lemma 7.2 ([16]) *A plane graph G is realizable as a Delaunay tessellation, with a given face f as the unbounded face and with a subset S of the boundary vertices of f as its extreme vertices, if and only if the graph G' obtained by inserting a new vertex v inside face f and connecting v to the vertices of S is inscribable. In particular, a plane graph G is realizable as a Delaunay triangulation, with a given face f as the unbounded face, if and only if the graph G' obtained from G by stellating f is simplicial and inscribable.*

Lemma 7.3 *The following properties hold.*

(a) Every 1-Hamiltonian, planar graph is inscribable [14].

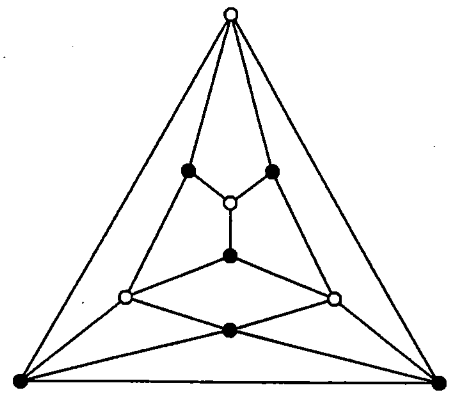

Figure 10: A noninscribable, self-dual graph with 10 vertices

(b) *Every inscribable graph is 1-tough [11].*

(c) *Every nonbipartite inscribable graph is 1-supertough [15].*

(d) *Every nonbipartite Delaunay tessellation is 1-tough [11].*

(e) *If G is inscribable and nonbipartite, any graph obtained from G by connecting two nonadjacent vertices on a common face is inscribable [16].*

Table 11 contains the number of noncircumscribable and noninscribable simplicial polyhedra for small values of n. (By duality, these numbers equal the number of inscribable and circumscribable trivalent polyhedra with $2n-4$ vertices). Both classes of polyhedra were computed by applying filters to the collection of simplicial polyhedra. The polyhedra were tested for circumscribability using the linear-time algorithm of [15]. The simplicial noninscribable polyhedra were computed using the following "triage" procedure. By Lemma 7.3(a), any 1-Hamiltonian polyhedron is inscribable. By Lemma 7.3(c), any simplicial polyhedron that fails to be 1-supertough is noninscribable. The remaining polyhedra (i.e., those that are 1-supertough but not 1-Hamiltonian) were then tested using an algorithm due to Igor Rivin, based on Lemma 7.1 (see [37] for details). The counts for 1-supertough and non-1-Hamiltonian simplicial polyhedra are also included in Table 11. Notice that all 1-supertough simplicial polyhedra with up to 14 vertices are also inscribable. However, the 15-vertex, 1-supertough graph of Figure 8(b) is not inscribable; for an explicit proof of this fact, see [15]). Smith has given bounds on the number of inscribable and circumscribable simplicial polyhedra [40].

The smallest noninscribable simplicial polyhedron is the fully stellated tetrahedron, and the smallest noncircumscribable polyhedron is the dual of the "clipped cube" (the polyhedron obtained by slicing off a corner of the cube with a plane, turning it into a triangle). The smallest polyhedron that is neither inscribable nor circumscribable is the self-dual polyhedron shown in Figure 10. This polyhedron is noninscribable because it is not 1-supertough, as can be seen by deleting the 4 white vertices.

n	$\|\mathcal{S}_n\|$	Not Circ	Not 1-Ham	Not 1ST	1ST, Not 1-Ham	Not Inscr
4	1	0	0	0	0	0
5	1	0	0	0	0	0
6	2	0	0	0	0	0
7	5	1	0	0	0	0
8	14	2	1	1	0	1
9	50	8	1	1	0	1
10	233	35	10	9	1	9
11	1,249	168	53	48	5	48
12	7,595	999	383	343	40	343
13	49,566	6,340	2,809	2,466	343	2,466
14	339,722	43,133	21,884	18,905	2,979	18,905
15	2,406,841	305,271				
16	17,490,241	2,231,377				

Table 11: The number of noncircumscribable and noninscribable simplicial polyhedra with n vertices.

8 Minimal non-Hamiltonian Delaunay Triangulations and Tessellations

The question of whether all nondegenerate Delaunay triangulations are Hamiltonian was posed in [30], [32] and, in a closely related form, in [39]. Counterexamples are known [9, 10, 28]. Here we discuss their minimality, and present (new) minimal counterexamples under the additional assumption of 3-connectivity.

The smallest non-Hamiltonian graph realizable as a Delaunay tessellation is the graph obtained by deleting a vertex from the cube; its minimality follows from the fact that the cube is the smallest non-1-Hamiltonian polyhedron. This example first appeared in [28]. The smallest non-Hamiltonian Delaunay triangulation is the example of [9]. This graph may be obtained by deleting one of the degree-7 vertices from the graph in Figure 6. The minimality of the example of [9] can be argued as follows: any non-Hamiltonian Delaunay triangulation must have the property that when its outer face is stellated it is simplicial, 1-supertough, and not 1-Hamiltonian. The graph of Figure 6 is the smallest graph with all these properties.

Both the preceding examples fail to be 3-connected. A 3-connected, non-Hamiltonian Delaunay triangulation with 25 vertices was constructed in [10]. This example is not minimal.

There are three polyhedral graphs with 13 vertices and 10 faces that can be realized as Delaunay tessellations. One of these is the graph of Figure 7(a). The other two are the two bipartite graphs shown in Figure 11. Note that here, and throughout this section,

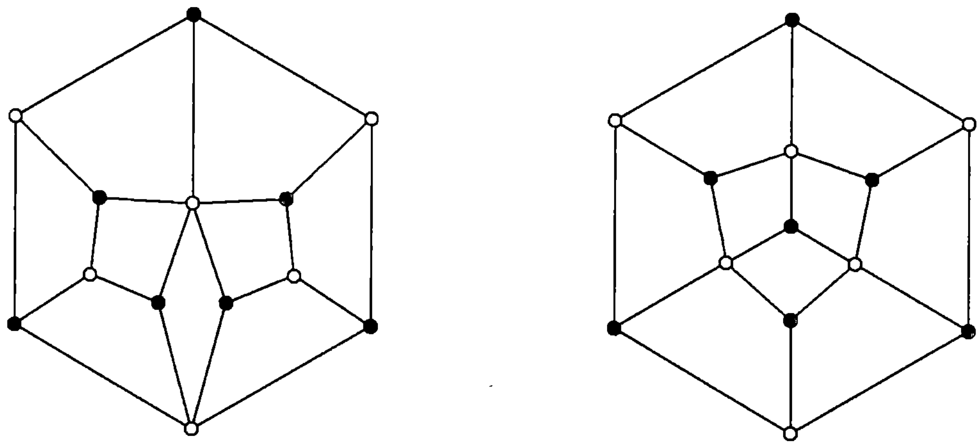

Figure 11: The two mininal non-Hamiltonian, 3-connected, bipartite graphs that are realizable as Delaunay Tessellations

the drawing is *not* a Delaunay tessellation; rather, it is a drawing of a graph that has a combinatorially equivalent realization as a Delaunay tessellation.

The graph of Figure 7(a) may be realized with either the face marked a or the face marked b as its outer face. Indeed, there are five different ways to fully or partially stellate a face of Figure 7(a) to get an inscribable polyhedron: fully stellate a, fully stellate b, or partially stellate either face a or b, connecting the new vertex to two dark vertices and one white one (this may be done in two different ways with face a, but only one with face b).

The minimality of these three examples may be argued as follows. If there exists a smaller non-Hamiltonian, 3-connected, Delaunay tessellation, it must be either bipartite or nonbipartite. If it is nonbipartite, then it is 1-tough by Lemma 7.3(d), but we saw in Section 6 that the smallest 1-tough non-Hamiltonian polyhedron is Figure 7(a). If it is bipartite, then let G' be a bipartite polyhedron obtained by partially stellating the outer face. G' must be 1-tough, bipartite, and have the property that removing some vertex leaves a 3-connected graph. A computer scan of the bipartite polyhedra with up to 14 vertices shows that there are exactly two bipartite polyhedra with these properties in that range, namely the two bipartite polyhedra obtained by partially stellating the outer faces of the graphs in Figure 11 with white vertices.

There are exactly two non-Hamiltonian, 3-connected graphs with 13 vertices that are realizable as Delaunay triangulations. These are shown in Figure 12. The graph G' obtained by stellating the outer face of a non-Hamiltonian, 3-connected Delaunay triangulation must be simplicial, 1-supertough, and have the property that removing some vertex leaves a 3-connected, non-Hamiltonian graph (so, in particular, G' must be non-1-Hamiltonian). The only two simplicial polyhedra that have these properties and no more than 14 vertices are the two polyhedra obtained by stellating the outer faces of the graphs in Figure 12.

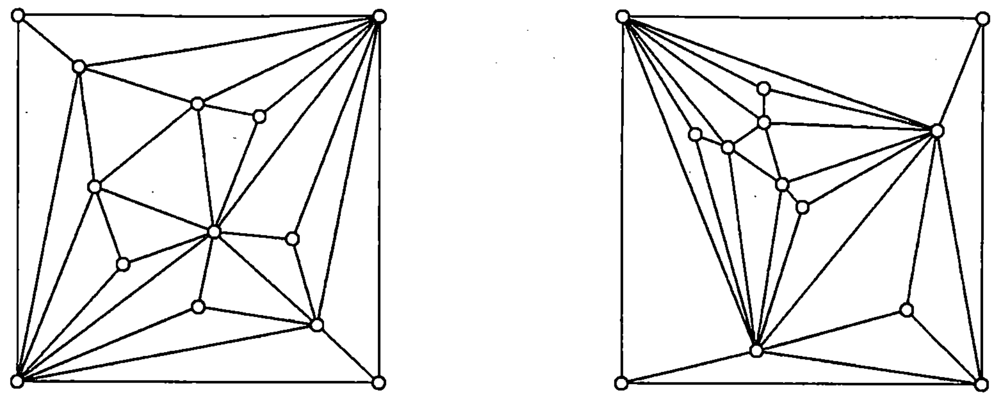

Figure 12: The two mininal non-Hamiltonian, 3-connected triangulations that are realizable as Delaunay Triangulations (13 vertices, 21 faces).

9 . Small Minimal non-Hamiltonian inscribable polyhedra

A 25-vertex, non-Hamiltonian, inscribable simplicial polyhedron was constructed in [10]. It follows from Lemma 7.3(a) that Thomassen's example of a 105-vertex planar graph that is 1-Hamiltonian but not Hamiltonian [41] represents an earlier discovery of an inscribable polyhedron, Here we present improved lower and upper bounds for the size of minimal non-Hamiltonian inscribable polyhedra, both in general and in the simplicial case. We deal with the unrestricted case first.

There are three non-Hamiltonian, inscribable polyhedra with 19 vertices and 13 polyhedra (Figure 13). We have verified that there are no non-Hamiltonian inscribable polyhedra for any other value of n and k that has a nonempty entry in Table 10. We have also verified that there is no non-Hamiltonian bipartite inscribable polyhedra with 22 or fewer vertices. Indeed, none of the bipartite polyhedra enumerated while constructing Table 7 are both 1-Hamiltonian and 1-tough. In fact, the smallest non-Hamiltonian bipartite polyhedron in which the two vertex sets in the bipartition have equal cardinalities has

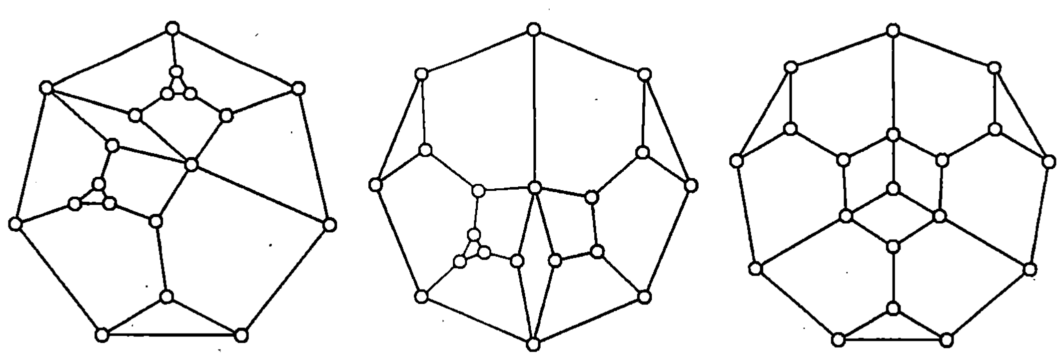

Figure 13: The three non-Hamiltonian, inscribable polyhedra with 19 vertices and 13 faces.

20 vertices. We conjecture that the three examples of Figure 13 are indeed minimal.

For the remainder of this section, let N denote the number of vertices in the smallest non-Hamiltonian, inscribable, simplicial polyhedron. We have determined that $18 \leq N \leq 20$, and we conjecture that the true answer is either 19 or 20.

The bound $N \leq 20$ holds because there are at least 11 nonisomorphic non-Hamiltonian, inscribable, simplicial polyhedra with 20 vertices. They are shown in Figure 14. They were constructed by looking for simplicial polyhedra that could be "pasted" together with the 9-vertex "building block" of Figure 9(b) (which we temporarily call T_9) to obtain a non-Hamiltonian inscribable polyhedron, in the same way that pasting T_9 together with itself creates the non-Hamiltonian, 1-supertough, noninscribable graph of Figure 8(b). Let K be such a simplicial polyhedron, with f the boundary of the face to be pasted together with the T_9. Assume that the orientation is such that f is the outer face of K. It can be shown that K must have the following properties (i.e., they are necessary, but perhaps not sufficient):

1. K must have the property that for any two vertices of f, any path between the two vertices that visits all vertices inside f must also visit the third vertex of f. (Otherwise, the pasted graph would be Hamiltonian.)

2. K must have the property that when f is stellated, the resulting graph is 1-supertough. (Otherwise, the pasted graph would not be 1-supertough.)

3. Define a Delaunay labeling of a simplicial polyhedron to be a labeling of the interior angles so that (1) the angles about each interior vertex sum to 360, (2) the angles about each triangle sum to 180, (3) all angles are positive, and (4) the sum of two angles facing a common edge is less than 180. K must have a Delaunay labeling in which the 3 angles facing the 3 outer edges have a total value less than 450.

We filtered $\mathcal{S}_n$ for polyhedra with these properties. We found none with 13 or fewer vertices, but 11 with 14 vertices. Each of the 20-vertex polyhedra obtained by pasting these 14-vertex polyhedra with T_9, as discussed above, is, indeed, non-Hamiltonian, inscribable, and simplicial. These 11 simplicial polyhedra are shown in Figure 14.

The bound $N > 17$ was determined by examining the non-Hamiltonian, 1-supertough, simplicial polyhedra with $n \leq 17$ (see column 5 of Table 9) and verifying that none were inscribable. The catalog of non-Hamiltonian simplicial polyhedra with 17 vertices was used to generate the imprimitive non-Hamiltonian, 1-supertough simplicial polyhedra with 18 vertices. There were 698 of these, none of which were inscribable. So if Conjecture 5.1 is true, then $N > 18$.

We conclude with one more collection of relevant counterexamples. We generated, for each $k \geq 13$, all non-Hamiltonian $(19, k)$-polyhedra that could be obtained by starting with the three graphs of Figure 13 and applying sequences of face-splitting operations. By Lemma 7.3(e), all polyhedra obtained in this way are inscribable. This process ultimately produced the six non-Hamiltonian inscribable $(19, 33)$-polyhedra shown in Figure 15. These polyhedra are inscribable triangulations, but they have one quadrangular face so they are not simplicial. In each case, adding a diagonal to the outer face (to make them

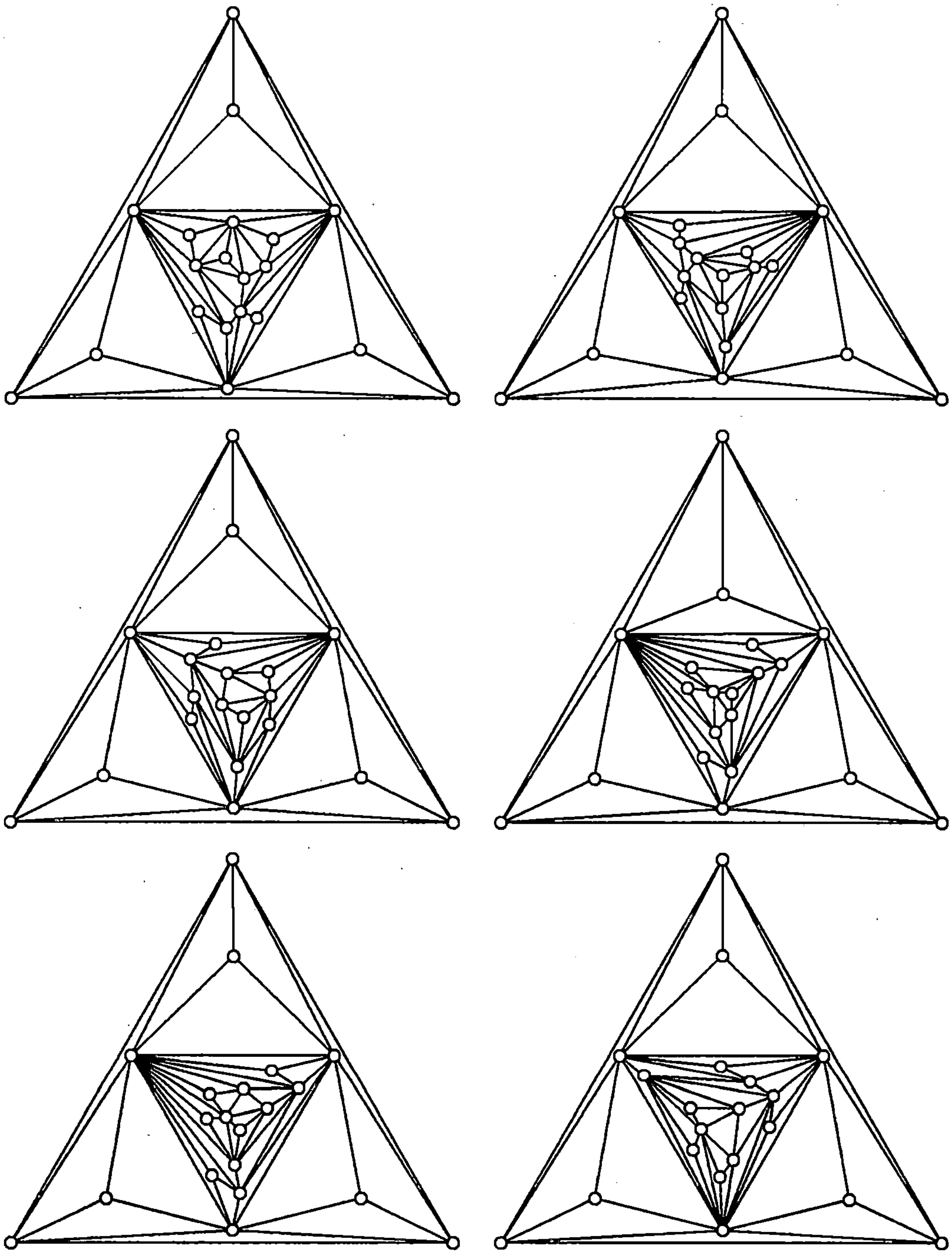

Figure 14: Eleven non-Hamiltonian, inscribable, simplicial polyhedra with 20 vertices [Part 1 of 2].

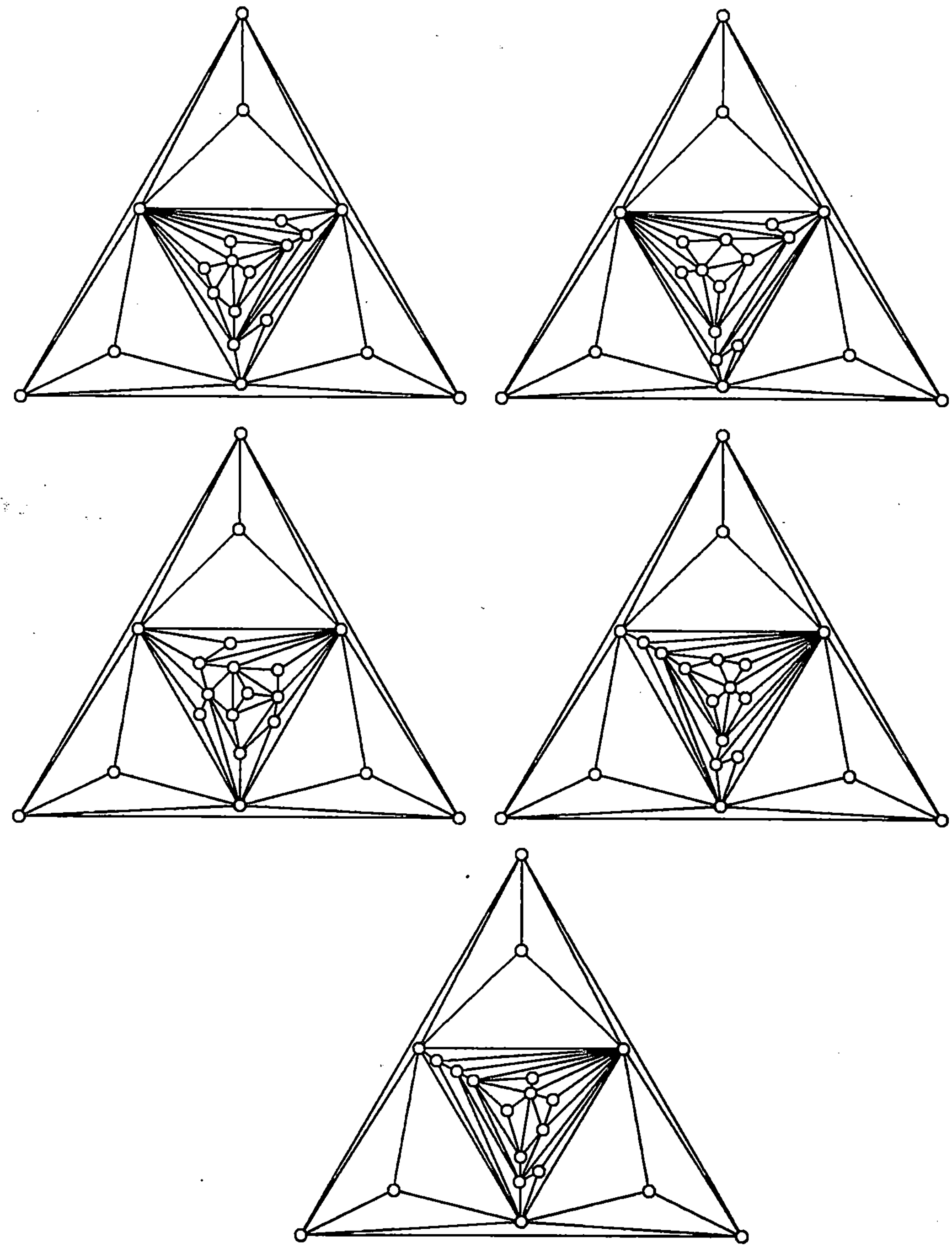

Figure 14: Eleven non-Hamiltonian, inscribable, simplicial polyhedra with 20 vertices [Part 2 of 2].

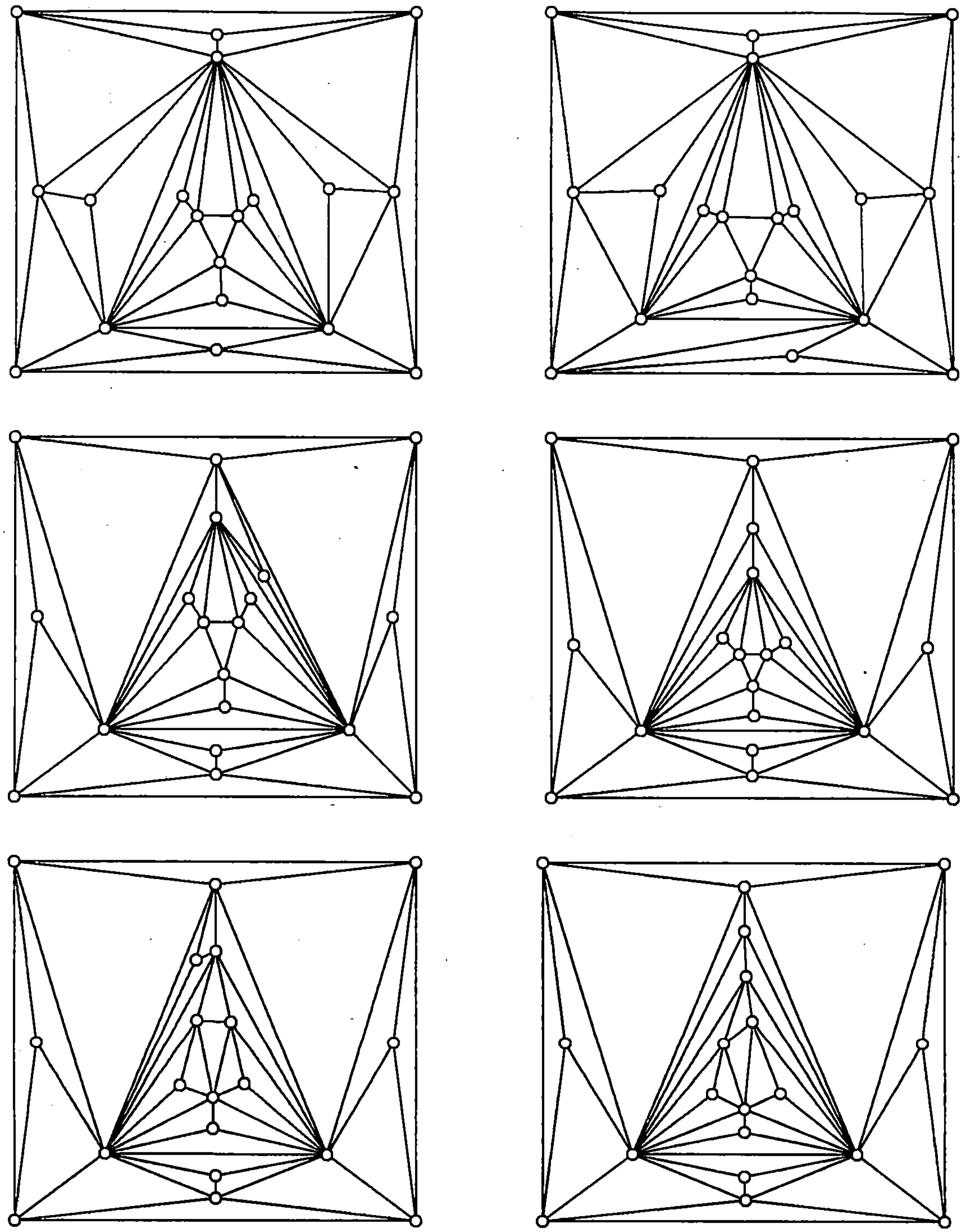

Figure 15: Six non-Hamiltonian, inscribable, polyhedra with 19 vertices and 33 faces.

simplicial) also makes them Hamiltonian. These examples justify the remarks made at the end of Section 5.

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